Poincaré series and rational cohomology rings of Kac-Moody groups and their flag manifolds*

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Abstract

In this paper, we study the rational cohomology rings of indefinite Kac-Moody groups and their flag manifolds. By extracting the information of cohomology from the Poincaré series, we are able to determine the rational cohomology rings of Kac-Moody groups and their flag manifolds. Since Kac-Moody groups and their flag manifolds are rational formal, we also determine their rational homotopy groups and rational homotopy types.

1 Introduction

Let $A = (a_{ij})$ be an $n \times n$ integer matrix satisfying

(1) For each $i, a_{ii} = 2$;

(2) For $i \neq j, a_{ij} \leq 0$;

(3) If $a_{ij} = 0$, then $a_{ji} = 0$.

then A is called a Cartan matrix.

By the work of Kac[15] and Moody[28] it is well known that for each $n \times n$ Cartan matrix A, there is a Lie algebra $g(A)$ associated to A which is called Kac-Moody Lie algebra. Then Kac and Peterson[17][18][19] constructed the Kac-Moody group $G(A)$ corresponding to the Lie algebra $g(A)$. In this paper we consider the quotient Lie algebra of $g(A)$ modulo its center $c(g(A))$ and the associated simply connected group $G(A)$ modulo $C(G(A))$, for convenience we still use the symbols $g(A)$ and $G(A)$.

Kac-Moody Lie algebras and Kac-Moody groups are divided into three types.

(1) Finite type, if A is positive definite. In this case, $G(A)$ is just the simply connected complex semisimple Lie group with Cartan matrix A.

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(2) Affine type, if A is positive semi-definite and has rank $n-1$.

(3) Indefinite type otherwise.

Denote the Weyl group of $g(A)$ by $W(A)$, then

$$W(A) = <\sigma_1, \cdots, \sigma_n | \sigma_i^2 = 1, 1 \le i \le n; (\sigma_i\sigma_j)^{m_{ij}} = 1, 1 \le i < j \le n> .$$

where $\sigma_1, \sigma_2, \cdots, \sigma_n$ are the Weyl reflections with respect to n simple roots $\alpha_1, \alpha_2, \cdots, \alpha_n$ of $g(A)$, $m_{ij} = 2, 3, 4, 6$ and ∞ as $a_{ij}a_{ji} = 0, 1, 2, 3$ and ≥ 4 respectively.

Each element $w \in W(A)$ has a decomposition of the form $w = \sigma_{i_1} \cdots \sigma_{i_k}, 1 \le i_1, \cdots, i_k \le n$. The length of w is defined as the least integer k in all of those decompositions of w, denoted by $l(w)$. The Poincaré series of $g(A)$ is the power series $P_A(q) = \sum\limits_{w \in W(A)} q^{2l(w)}$.

For the Kac-Moody Lie algebra $g(A)$, there is the Cartan decomposition $g(A) = h \oplus \sum\limits_{\alpha \in \Delta} g_\alpha$, where h is the Cartan sub-algebra and Δ is the root system of $g(A)$. Let $b = h \oplus \sum\limits_{\alpha \in \Delta^+} g_\alpha$ be the Borel sub-algebra, then b corresponds to a Borel subgroup $B(A)$ in the Kac-Moody group $G(A)$. The homogeneous space $F(A) = G(A)/B(A)$ is called the flag manifold of $G(A)$. By Kumar[26], $F(A)$ is an ind-variety.

The flag manifold $F(A)$ admits a CW-decomposition of Schubert cells which are indexed by the elements of Weyl group $W(A)$. For each $w \in W(A)$, the real dimension of Schubert variety X_w is $2l(w)$. So the Poincaré series of flag manifold $F(A)$ is just the Poincaré series $P_A(q)$ of $g(A)$.

By the well known results about the cohomology and Poincaré series of flag manifolds of Kac-Moody groups, for example see Kichiloo[23] and Kumar[26], it follows that: the rational cohomology rings $H^*(G(A))$ and $H^*(F(A))$ are locally finite. Hence they are generated by countable number of generators. Steinberg[32] proved that the Poincaré series $P_A(q)$ of the Kac-Moody flag manifold $F(A)$ is a rational function.

The rational cohomology rings of Kac-Moody groups and their flag manifolds of finite or affine type are extensively studied by many algebraic topologists. For reference, see Pontrjagin[30], Hopf[11], Borel[2][3][4], Bott and Samelson[5], Bott[6], Milnor and Moore[27], Chevalley[8], Bernstein, Gel'fand and Gelfand[1], Demazure[9], Kostant and Kumar[24] and Kichiloo[23] etc.

Below is the well known theorem about the rational cohomology of Hopf spaces.

Theorem(Hopf): Let G be a connected H-space which has the homotopy type of a CW-complex, then the rational cohomology rings $H^*(G)$ is a Hopf algebra and as algebra it is isomorphic to the the tensor product of a polynomial algebra $P(V_0)$ and a exterior algebra $\Lambda(V_1)$, where V_0 and V_1 are respectively the set of even and odd degree free generators of $H^*(G)$.

For a Kac-Moody group $G(A)$, $H^k(G(A))$ is a finite dimension rational vector space for each $k \ge 0$. Denote the number of degree k generators in $V = V_0 \cup V_1$ by i_k, then the Poincaré series of $G(A)$ is

$$P_G(q) = \prod_{k=1}^{\infty} \frac{(1-q^{2k-1})^{i_{2k-1}}}{(1-q^{2k})^{i_{2k}}}. \tag{1}$$

An important result is

Theorem 1: The cohomology ring $H^*(G(A))$ is determined by its Poincaré series.

The Poincaré series for finite and affine type Kac-Moody groups and their flag manifolds are known. The computations of Poincaré series for hyperbolic Kac-Moody flag manifolds are discussed

in Gungormez and Karadayi[10], Chapovalov, Leites and Stekolshchik[7]. The general indefinite case is considered by the authors in [13][14].

But for rational cohomology rings of indefinite Kac-Moody groups and their flag manifolds, little is known. In fact except for $n \leq 2$(see [23]), not a single case is computed. So a natural question is

Problem: Compute the rational cohomology rings for Kac-Moody groups and their flag manifolds of indefinite type.

In this paper our main goal is to solve this problem. Since Kac-Moody groups and their flag manifolds are rational formal[31], their rational homotopy types are determined by their rational cohomology rings. Our strategy is:

(1) Compute the Poincaré series $P_A(q)$ of $F(A)$ from the Poincaré series of $G(A)$ by applying Leray-Serre spectral sequence;

(2) Compute the Poincaré series $P_A(q)$ of $F(A)$ from the Cartan matrix A by using the method in [13][14];

(3) Determine $i_k, k \geq 1$ in Equation (1) by comparing the Poincaré series obtained in (1) and (2).

Let the classifying map of principal $B(A)$-bundle $\pi : G(A) \to F(A)$ be $j : F(A) \to BB(A)$, where $BB(A)$ denotes the classifying space of Borel subgroup $B(A)$. There is a homotopy fibration $G(A) \xrightarrow{\pi} F(A) \xrightarrow{j} BB(A)$. By analyzing the Leray-Serre spectral sequence, we show

Theorem 2: For a Kac-Moody group $G(A)$ with Poincaré series $P_G(q)$ as in equation (1), the Poincaré series of $F(A)$ is

$$P_A(q) = \frac{\prod_{k=1}^{\infty}(1-q^{2k})^{i_{2k-1}}}{(1-q^2)^n} \frac{1}{\prod_{k=1}^{\infty}(1-q^{2k})^{i_{2k}}}. \tag{2}$$

For a Kac-Moody group $G(A)$, it is shown by Kac[20] that the set V_1 is a finite set and the number l of elements in V_1 is less or equal to n. So the exterior algebra part of $H^*(G(A))$ is of finite dimension.

Theorem 3: Let $P_A(q)$ be the Poincaré series of flag manifold $F(A)$, then the sequence $i_2 - i_1, i_4 - i_3, \cdots, i_{2k} - i_{2k-1}, \cdots$ can be derived from $P_A(q)$. In fact $P_A(q)$ can also be recovered from the sequence $i_2 - i_1, i_4 - i_3, \cdots, i_{2k} - i_{2k-1}, \cdots$.

But to determine the rational homotopy type and rational cohomology of $G(A)$, we need to determine the sequence $i_1, i_2, \cdots, i_k, \cdots$. So except for the Poincaré series $P_A(q)$, more ingredients are needed. If we can determine all the degrees of elements in V_1, then we will determine the sequence $i_1, i_3, \cdots, i_{2k-1}, \cdots$. Combining with Theorem 3, we also determine the sequence $i_2, i_4, \cdots, i_{2k}, \cdots$, hence work out the rational cohomology ring and rational homotopy type of $G(A)$.

By Kac[20], Kac and Peterson[21] the sequence $i_1, i_3, \cdots, i_{2k-1}, \cdots$ can be determined by the ring of polynomial invariants of Weyl group $W(A)$. In fact i_{2k-1} is just the number of degree k basic invariant polynomials of $W(A)$.

In [34] the authors prove the following result.

Theorem 4: Let A be an indecomposable and indefinite Cartan matrix. If A is symmetrizable, then $I(A) = \mathbb{Q}[\psi]$; If A is non symmetrizable, then $I(A) = \mathbb{Q}$. Where $I(A)$ is the ring of polynomial invariants of Weyl group $W(A)$, and ψ is a invariant bilinear form on $g(A)$.

This theorem is a generalization of Moody's result(see [29]) for symmetrizable hyperbolic Cartan matrices.

By Theorem 4, for an indecomposable and indefinite Cartan matrix A, $i_{2k-1} = 0$ for all $k > 0$ except for $k = 2$. And for $k = 2$, if A is symmetrizable, $i_3 = 1$; If A is non symmetrizable, $i_3 = 0$. Hence the sequence $i_1, i_3, \cdots, i_{2k-1}, \cdots$ is worked out. Combining with Theorem 3 we eventually determine the sequence $i_1, i_2, \cdots, i_k, \cdots$. As a consequence the rational cohomology and rational homotopy types of the Kac-Moody group $G(A)$ and its flag manifold $F(A)$ are determined.

This paper is organized as below. In section 2, we introduce some algebraic and combinatorial results we needed. In section 3 we discuss the decompositions of order of Weyl group $W(A)$ and Poincaré series of flag manifold $F(A)$. In section 4 we compute the Poincaré series $P_A(q)$ of flag manifold $F(A)$ from the Poincaré series $P_G(q)$ of $G(A)$ via Leray-Serre spectral sequence. The main theorem of this paper will be concluded in Section 5. And an example is given in section 6.

2 Some algebras and combinatorics

Let $\mathbb{Z}_1[q]$ be the set of power series of q with integer coefficients and the constant item 1. That is for $f(q) \in \mathbb{Z}_1[q]$,

$$f(q) = 1 + a_1 q + a_2 q^2 + \cdots + a_k q^k + \cdots, a_k \in \mathbb{Z}, k > 0.$$

We have the following result.

Proposition 1: $f(q) \in \mathbb{Z}_1[q]$ can be expanded uniquely into the form of products $\prod_{k=1}^{\infty}(1-q^k)^{i_k}$, where $i_1, i_2, \cdots, i_k, \cdots$ are integer sequence.

Proof: For $f(q)$, we define an integer sequence $i_1, i_2, \cdots, i_k, \cdots$ inductively.

At first we set $i_1 = a_1$. Let $f^{(1)}(q) = f(q)/(1-q)^{i_1}$, then $f^{(1)} \in \mathbb{Z}_1[q]$ and

$$f^{(1)}(q) = 1 + a_2^{(1)} q^2 + a_3^{(1)} q^3 + \cdots + a_k^{(1)} q^k + \cdots.$$

Set $i_2 = a_2^{(1)}$ and let $f^{(2)}(q) = f^{(1)}(q)/(1-q^2)^{i_2}$, then $f^{(2)} \in \mathbb{Z}_1[q]$ and

$$f^{(2)}(q) = 1 + a_3^{(2)} q^3 + a_4^{(2)} q^4 + \cdots + a_k^{(2)} q^k + \cdots.$$

Set $i_3 = a_3^{(2)}$ and continue the same procedure we get integer sequence $i_1, i_2, \cdots, i_k, \cdots$.

By checking the previous procedure and comparing the coefficients of the power series concerned, we prove the uniqueness of the sequence $i_1, i_2, \cdots, i_k, \cdots$.

We call the sequence $i_1, i_2, \cdots, i_k, \cdots$ the characteristic sequence of power series $f(q)$. By Proposition 1, there is a one to one correspondence between $\mathbb{Z}_1[q]$ and the set of integer sequences indexed on the set $\mathbb{N}$ of natural numbers. And if $f_1(q), f_2(q)$ correspond to the integer sequences $i_1, i_2, \cdots, i_k, \cdots$ and $j_1, j_2, \cdots, j_k, \cdots$, then the product $f_1(q)f_2(q)$ corresponds to integer sequence $i_1 + j_1, i_2 + j_2, \cdots, i_k + j_k, \cdots$.

The Poincaré series $P_A(q)$ of $F(A)$ is of the form $P_A(q) = G(q^2)$. By applying Proposition 1 to $G(q)$ we will prove Theorem 1.

The characteristic sequence $i_1, i_2, \cdots, i_k, \cdots$ can be computed by using Möbius Inversion Theorem.

Proposition 2: Let $f(q) \in \mathbb{Z}_1[q]$, suppose $\ln(f(q)) = b_1 q + \frac{b_2}{2}q^2 \cdots + \frac{b_k}{k}q^k + \cdots$, then

$$i_k = \frac{1}{k}\sum_{n|k} \mu(n) b_{k/n}.$$

Proof: For

$$f(q) = \prod_{k=1}^{\infty}(1-q^k)^{i_k}$$

take "ln" to the two sides of equation $f(q) = \prod_{k=1}^{\infty}(1-q^k)^{i_k}$, we get

$$b_1 q + \frac{b_2}{2}q^2 \cdots + \frac{b_k}{k}q^k + \cdots = \ln \prod_{k=1}^{\infty}(1-q^k)^{i_k}$$

the right side$= \sum_{k=1}^{\infty} i_k \ln(1-q^k) = \sum_{k=1}^{\infty} i_k \sum_{n=1}^{\infty} \frac{q^{kn}}{n} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{i_k}{n} q^{kn} = \sum_{n=1}^{\infty} (\sum_{k|n} \frac{k \cdot i_k}{n}) q^n$

By comparing the coefficients of the two sides we get $\sum_{k|n} k \cdot i_k = b_n$

We need the following theorem to compute i_k.

Möbius Inversion Theorem: Let $F(n), f(n)$ be two sequences indexed on natural numbers, if $F(n) = \sum_{k|n} f(k)$, then $f(k) = \sum_{n|k} \mu(n) F(k/n)$, where $\mu(n)$ satisfies:

(1) When $n = 1, \mu(n) = 1$.

(2) When $n = \prod_{i=1}^{r} p_i$, $\mu(n) = (-1)^r$, where p_i's are different prime integers.

(3) $\mu(n) = 0$ otherwise.

By the Möbius Inversion Theorem, from $\sum_{k|n} k \cdot i_k = b_n$ we get

$$k \cdot i_k = \sum_{n|k} \mu(n) b_{k/n}$$

Example 1: If $f(q) = 1 - 2q$, then in this case, $b_n = 2^n$, so

$$k \cdot i_k = \sum_{d|k} \mu(d) 2^{\frac{k}{d}}.$$

Take $k = 18$, all the divisors of 18 is $1, 2, 3, 6, 9, 18$ and

$$\mu(1) = 1, \mu(2) = -1, \mu(3) = -1, \mu(6) = 1, \mu(9) = \mu(18) = 0,$$

then $18 i_{18} = 2^{18} - 2^9 - 2^6 + 2^3$, so $i_{18} = 14532$.

We list the computation results as below for $1 \leq k \leq 18$.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
i_k	2	1	2	3	6	9	18	30	56	99	186	335	630	1161	2182	4080	7710	14532

3 Decompositions of Weyl groups and Poincaré series

Let $G(A)$ be a Kac-Moody group and $W(A)$ be its Weyl group. From the definition of Poincaré series $P_A(q)$, we see as q approaches to 1, $P_A(q)$ tends to $|W(A)|$(the order of $W(A)$). Hence the Poincaré series $P_A(q) = \sum\limits_{w\in W(A)} q^{2l(w)}$ can be regarded as deformation(or quantization) of $|W(A)|$.

For classical Lie group of type A_n, $G(A_n) = \mathrm{SL}(n+1,\mathbb{C})$. The Weyl group of $G(A_n)$ is $W(A_n) \cong S_{n+1}$, the permutation group of set $\{1, 2, \cdots, n, n+1\}$, $|W(A_n)| = (n+1)!$. The Poincaré series of flag manifold $F(A_n) = \mathrm{SL}(n+1,\mathbb{C})/B$ is $P_{A_n}(q) = \prod\limits_{i=1}^{n+1} \dfrac{1-q^{2k}}{1-q^2}$. If we set $[k] = \dfrac{1-q^{2k}}{1-q^2}$, then as q approaches to 1, $[k]$ approaches to k. The Poincaré series is $[n+1]! = [n+1][n]\cdots[1]$.

In fact for the Kac-Moody group $G(A)$ of finite type, if

$$H^*(G(A)) \cong \Lambda(x_1, x_2, \cdots, x_n), \deg x_k = 2d_k - 1,$$

then $|W(A)| = \prod\limits_{k=1}^{n} d_k$ and $P_A(q) = \prod\limits_{k=1}^{n} [d_k]$, where d_k's are the degrees of the basic invariant polynomials of $W(A)$.

For a Kac-Moody group $G(A)$ of affine or indefinite type, $|W(A)|$ is infinite, so at first sight the decomposition of $|W(A)|$ is meaningless. But if we consider $P_A(q)$, through a regularization procedure as in quantum fields theory, we can give an interesting decomposition.

Let $G(\widetilde{A})$ be an untwisted affine Kac-Moody groups with $\widetilde{A}$ the extended Cartan matrix of a Cartan matrix A of finite type. For the flag manifold $F(\widetilde{A})$, the Poincaré series is

$$P_{\widetilde{A}}(q) = P_A(q) \prod_{i=1}^{n} \frac{1}{1-q^{2d_k-2}} = \prod_{k=1}^{n} [d_k][\infty]_{d_k-1}.$$

where $[\infty]_k = \frac{1}{1-q^{2k}}$.

So we have the decomposition $|W(\widetilde{A})| = \prod\limits_{k=1}^{n} d_k \prod\limits_{k=1}^{n} \infty_{d_k-1}$.

For the Weyl groups and Poincaré series of twisted affine flag manifolds, the computation results are almost the same as the twisted case. The decomposition of order of Weyl groups and Poincaré series for indefinite case can also be done. But due to the relation $[k] = [\infty]_1[\infty]_k^{-1}$, the decomposition is not unique. For details see [14].

4 Leray-Serre spectral sequence

In this section we use the Leray-Serre spectral sequence of fibration $G(A) \xrightarrow{\pi} F(A) \xrightarrow{j} BB(A)$ to compute the cohomology and Poincaré series of $F(A)$ from those of $G(A)$. For reference see [33].

$BB(A)$ is homotopic to the n-fold Cartesian product of classifying space of $B\mathbb{C}^*(\mathbb{C}^* = \mathbb{C}-\{0\})$, denote the cohomology generators of $H^*(BB(A))$ by $\omega_1, \cdots, \omega_n, \deg\omega_i = 2$, the free generators of V_1 by $y_1, \cdots, y_l$, and the free generators of V_0 by $z_1, \cdots, z_k, \cdots$. Where $\omega_1, \cdots, \omega_n$ correspond to the fundamental dominant weights of $g(A)$. We have the spectral sequence $(E_r^{p,q}, d_r)$ with

$$E_2^{p,q} = H^p(BB(A); H^q(G(A))) \cong \mathbb{Q}[\omega_1, \cdots, \omega_n] \otimes \Lambda(y_1, \cdots, y_l) \otimes \mathbb{Q}[z_1, \cdots, z_k, \cdots].$$

The differential $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$ is given by $d_2(\omega_i \otimes 1) = 0, d_2(1 \otimes z_k) = 0$ and $d_2(1 \otimes y_j) = f_j \otimes 1, 1 \leq j \leq l$, where $f_j \in H^*(BB(A))$ is a polynomial of $\omega_i, 1 \leq i \leq n$. A routine computation shows

$$H^*(F(A)) \cong E_3^{*,*} \cong \mathbb{Q}[\omega_1, \cdots, \omega_n]/ < f_j, 1 \leq j \leq l > \otimes \mathbb{Q}[z_1, \cdots, z_k, \cdots].$$

By Kac[20], $f_1, \cdots, f_l$ is a regular sequence in $H^*(BB(A))$, so $l \leq n$. Therefore we conclude that the Poincaré series of $F(A)$ is given by the Formula (2). This proves the Theorem 2.

By simplifying the Formula (2), we get

$$P_A(q) = \frac{1}{(1-q^2)^{n+i_2-i_1}} \frac{1}{\prod_{k=2}^{\infty}(1-q^{2k})^{i_{2k}-i_{2k-1}}} \tag{3}$$

By using Proposition 1 to Equation (3), we prove the Theorem 3.

5 The rational homotopy type of $G(A)$

The Poincaré series $P_A(q)$ can be computed easily by an inductive procedure, see [13][14] for example. From $P_A(q)$ we can compute the sequence $i_2 - i_1, i_4 - i_3, \cdots, i_{2k} - i_{2k-1}, \cdots$. But to determine the rational homotopy type of $G(A)$, we need to determine the sequence $i_1, i_2, \cdots, i_k, \cdots$. So except for the Poincaré series $P_A(q)$, we need more ingredients. Theorem 4 fills the gap. For its proof, see [34].

By Theorem 4 for indecomposable and indefinite Cartan matrix A, $i_1, i_3, \cdots, i_{2k+1}, \cdots$ is determined, combining with Theorem 3, we determine the complete sequence $i_1, i_2, \cdots, i_k, \cdots$.

Since $G(A)$ is a simply connected group, we get $i_1 = 0$. By virtue of Schubert decomposition, we know that $H^2(F(A))$ is spanned by the Schubert classes corresponding to n Weyl reflections. So $n + i_2 - i_1 = n$, hence $i_2 = 0$. To determine i_3 we need the following definition which is well known in Kac-Moody Lie algebras theory.

Definition 1: An $n \times n$ Cartan matrix A is symmetrizable if there exist an invertible diagonal matrix D and a symmetric matrix B such that $A = DB$. $g(A)$ is called a symmetrizable Kac-Moody Lie algebra if its Cartan matrix is symmetrizable.

The symmetrizability of a Cartan matrix A is intimately related to the existence of non degenerate invariant bilinear form ψ on $g(A)$. From Theorem 4, it follows directly that If $g(A)$ is a symmetrizable Kac-Moody Lie algebra, then $i_3 = 1$, otherwise $i_3 = 0$. And $i_{2k-1} = 0$ for $k \geq 3$.

Set $\epsilon(A) = 1$ or 0 depending on A is symmetrizable or not as in [20], then we can state our main theorem as below.

Theorem 5: For an indecomposable and indefinite Cartan matrix A, if A is symmetrizable, then

$$H^*(G(A)) \cong \Lambda_{\mathbb{Q}}(y_3) \otimes \mathbb{Q}[z_1, \cdots, z_k, \cdots]$$

and

$$H^*(F(A)) \cong \mathbb{Q}[\omega_1, \cdots, \omega_n]/ < \psi > \otimes \mathbb{Q}[z_1, \cdots, z_k, \cdots].$$

If A is non symmetrizable, then

$$H^*(G(A)) \cong \mathbb{Q}[z_1, \cdots, z_k, \cdots]$$

and

$$H^*(F(A)) \cong \mathbb{Q}[\omega_1, \cdots, \omega_n] \otimes \mathbb{Q}[z_1, \cdots, z_k, \cdots].$$

where $\deg z_k \geq 4$ are even for all k and their degrees can be determined from the Poincaré series $P_A(q)$ and $\epsilon(A)$.

The rational homotopy groups and the rational minimal model of the Kac-Moody group $G(A)$ and its flag manifold $F(A)$ can be computed from this theorem easily.

Kumar[25] proved that for Kac-Moody Lie algebra $g(A)$, the Lie algebra cohomology $H^*(g(A), \mathbb{C}) \cong H^*(G(A)) \otimes \mathbb{C}$, so we also compute $H^*(g(A), \mathbb{C})$.

6 An example

We end this paper by the following example.

Example 2: Let A be a rank n Cartan matrix A which satisfies $a_{ij}a_{ji} \geq 4$ for all $i \neq j$, then the Weyl group of $G(A)$ is

$$W(A) = < \sigma_1, \cdots, \sigma_n | \sigma_i^2 = 1, 1 \leq i \leq n > .$$

The length k elements in $W(A)$ are in one to one correspondence to the words $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ satisfying $i_t \neq i_{t+1}, 1 \leq t \leq k-1$. So the number of length k elements in $W(A)$ is $n(n-1)^{k-1}$ and the Poincaré series is

$$P_A(q) = \sum_{k=0}^{\infty} n(n-1)^{k-1} q^{2k} = \frac{1+q^2}{1-(n-1)q^2}$$

By the well known Witt formula(see [22])

$$1 - nq = \prod_{k=1}^{\infty} (1-q^k)^{\dim L_n^k}$$

where L_n is the free graded Lie algebra generated by n elements with degree 1 and $L_n = \bigoplus_{k=1}^{\infty} L_n^k$. $\dim L_n^k$ is the dimension of the degree k homogeneous component of L_n. It is computed by using the Möbius Inversion Theorem. See Example 1 for the case $n = 2$. Hence

$$P_A(q) = \frac{1-q^4}{(1-q^2)^n} \frac{1}{\prod_{k=2}^{\infty} (1-q^{2k})^{\dim L_{n-1}^k}}.$$

From the expression of $P_A(q)$, we get:

If A is symmetrizable, then $i_1 = i_2 = 0, i_3 = 1$ and $i_{2k-1} = 0$ for $k \geq 3$, $i_{2k} = \dim L_{n-1}^k$ for $k \geq 2$.

If A is not symmetrizable, then $i_1 = i_2 = i_3 = 0, i_4 = \dim L_{n-1}^2 - 1$ and $i_{2k-1} = 0, i_{2k} = \dim L_{n-1}^k$ for $k \geq 3$.

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